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SU(1,1) symmetry of multimode squeezed states

Z Shaterzadeh-Yazdi, P S Turner and B C Sanders

Institute for Quantum Information Science, University of Calgary, 2500 University Drive NW, Calgary, Alberta T2N 1N4, Canada

E-mail: zyazdi@qis.ucalgary.ca and pturner@qis.ucalgary.ca

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Abstract

We show that a class of multimode optical transformations that employ linear optics plus two-mode squeezing can be expressed as SU(1,1) operators. These operations are relevant to state-of-the-art continuous variable quantum information experiments including quantum state sharing, quantum teleportation and multipartite entangled states. Using this SU(1,1) description of these transformations, we obtain a new basis for such transformations that lies in a useful representation of this group and lies outside the often-used restriction to Gaussian states. We analyze this basis, show its application to a class of transformations and discuss its extension to more general quantum optical networks.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The Heisenberg uncertainty principle ensures that the light is noisy at a quantum level. Mathematically, this noise arises because of non-commutativity of the sine and cosine quadratures of light. In the vacuum state, the noise in both quadratures are equal, but the noise can be squeezed—i.e. the noise in one quadrature can be reduced at the expense of increasing noise in the complementary quadrature—by propagation through nonlinear optical devices such as crystals and gases. In quantum information applications, squeezers, which shift fluctuations from one quadrature to another [1], provide an entanglement resource through correlated noise in two modes with these correlations stronger than anything possible in a classical description of light [2, 3].

The importance of squeezed light in quantum information is evident in its role in some of the most important experiments in quantum information science: teleportation [4], entanglement swapping [5], tests of local realism and Einstein–Podolsky–Rosen (EPR) paradox [6], and quantum state sharing [7, 8]. These continuous variable quantum information

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experiments involve more than two modes, and entanglement between more than two modes is inherent in these experiments.

Here we analyze multimode squeezed light. Specifically, we are interested in the case of two-mode squeezed light (which could be generated by a two-mode squeezer or, alternatively, by two single-mode squeezers in opposite phase with their outputs mixed at a beam splitter to yield two-mode squeezed light [9]), with the squeezed field distributed over multiple modes via linear optics to share the squeezed light between modes. Mathematically, the transformation is described by a matrix, which transforms input state vectors into output state vectors. The choice of basis can simplify the mathematical description as well as the calculations themselves. For example, if the inputs are Gaussian states, it is convenient to use a Gaussian basis, and transformations can be fully described in terms of the multivariate vector of means and the multivariate covariance matrix [10]. For general input states, the Fock number state basis is a useful basis to write general transformations; two alternative representations that transform simply under squeezing and linear optics are provided by the Wigner function [11, 12] and by the position representation [8]. The problem with these approaches for general input states is that the size of the matrix grows linearly in the number of modes.

On the other hand, the use of group theoretical concepts in quantum optics is becoming more and more prevalent [13–16]. Understanding the symmetry of a quantum system and identifying its group properties allow one to use a large class of mathematical tools to simplify problems [17]. Here, we show that an alternative basis can be obtained, which relies on determining the symmetries of the network transformation and using the powerful machinery of Lie group theory, including the Wigner–Eckart theorem [18]. In our basis, the mathematical description has a constant size independent of the number of modes. Our approach builds upon the SU(1,1) symmetry inherent in combining squeezing with linear optics in two modes.

In the absence of squeezing, n-mode linear optical transformations are elements of the compact special unitary group SU(n) [19]; with squeezers included in the network, optical network transformations are given by elements of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ [10, 20]. The fact that these transformations are the members of the symplectic group guarantees the simplicity of describing Gaussian state transformations using just covariance matrices and transformations of means [21], but this simplicity does not carry over to general non-Gaussian states.

Here, we show that a broad class of quantum optical networks involving two-mode squeezing and linear optics of arbitrarily many modes, can be greatly simplified by exploiting a $\mathrm{SU}(1,1) \cong \mathrm{Sp}(2, \mathbb{R}) \subset \mathrm{Sp}(2n, \mathbb{R})$ symmetry in such systems. This symmetry allows us to describe the transformation with fixed size independent of the number n of modes. All examples mentioned above—quantum teleportation, entanglement swapping, state sharing and tests of local realism—have transformations that are members of this class. We exploit this SU(1,1) symmetry by finding a basis that reduces the n-mode Fock states into irreducible representations (irreps) of this group.

In addition to finding the SU(1,1) symmetry for a broad class of interferometers comprising linear optics and two-mode squeezing, and finding a convenient basis in which the transformations have fixed size independent of the number n of modes, our work points to new directions in studies of more complex interferometers involving more squeezers. The Bloch–Messiah theorem applied to quantum optical networks [22, 23] is a powerful tool to reduce such interferometers to all linear optical transformations, followed by single-mode squeezers, followed by all linear optical transformations. The approach we establish here may provide an alternative to decomposing such interferometers by concatenating interferometers into a larger whole. We discuss this possible future direction for research in the conclusions.

2. Background: two-mode squeezing and the SU(1,1) Lie group

Two-mode squeezed states can be generated either by entangling two independent single-mode squeezed states via a 50:50 beamsplitter, or by employing the non-degenerate operation of a nonlinear medium in the presence of two incoming modes [24]. The unitary operator describing two-mode squeezing is

$$\hat{S}_{ab}(\eta) = \exp[-\mathrm{i}(\eta\hat{a}\hat{b} + \eta^*\hat{a}^\dagger\hat{b}^\dagger)/2], \tag{1}$$

with $\hat{a}, \hat{b}$ the annihilation operators of the incoming modes and $\eta \in \mathbb{C}$ the squeezing parameter. This operator gives a unitary representation of the SU(1,1) Lie group on the Hilbert space of two modes. As such, it is generated by a $\mathfrak{su}(1,1)$ Lie algebra given by

$$\hat{K}_+ = \hat{a}^\dagger\hat{b}^\dagger, \qquad \hat{K}_- = \hat{a}\hat{b}, \qquad \hat{K}_0 = \tfrac{1}{2}(\hat{a}^\dagger\hat{a} + \hat{b}\hat{b}^\dagger), \tag{2}$$

which satisfy the commutation relations

$$[\hat{K}_0, \hat{K}_\pm] = \pm\hat{K}_\pm, \qquad [\hat{K}_-, \hat{K}_+] = 2\hat{K}_0. \tag{3}$$

The SU(1,1) Casimir invariant is

$$\hat{K}^2 = \hat{K}_0^2 - \tfrac{1}{2}(\hat{K}_+\hat{K}_- + \hat{K}_-\hat{K}_+) = \tfrac{1}{4}[(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})^2 - \hat{\mathbb{1}}]. \tag{4}$$

Eigenvalues of $\hat{K}^2$ are used to label the irreps of SU(1,1), and eigenvalues of $\hat{K}_0$ provide an index for a basis of each irrep. Denoting such an orthonormal basis by $\{|k,\mu\rangle\}$, we have the following SU(1,1) action:

$$\begin{aligned}
\hat{K}_\pm|k,\mu\rangle &= \sqrt{(\mu \pm k)(\mu \mp k \pm 1)}|k,\mu \pm 1\rangle, \\
\hat{K}_0|k,\mu\rangle &= \mu|k,\mu\rangle, \\
\hat{K}^2|k,\mu\rangle &= k(k-1)|k,\mu\rangle.
\end{aligned} \tag{5}$$

Due to the non-compactness of this group, all unitary irreps are infinite dimensional. There are in fact several different series of irreducible representations of SU(1,1) distinguished by the domains of these eigenvalues [17]. For now, we are only interested in the usual positive discrete series of irreps where k is a non-negative half integer and μ takes values $k+m$ for $m = 0, 1, 2, \ldots$ carried by a Hilbert space denoted by $\mathscr{D}_k^+$.

If we label the number of excitations for modes a and b by n_a and n_b, respectively, then k satisfies

$$k = \frac{|n_a - n_b| + 1}{2}. \tag{6}$$

As the relabeling of modes $a \leftrightarrow b$ is physically equivalent to the original labeling, we consider the cases $\pm(n_a - n_b)$ to be equivalent irreps (hence the absolute value above). Thus, we say that each irrep $k > 0$ occurs twice, and the Hilbert space decomposes as

$$\mathscr{H}_a \otimes \mathscr{H}_b = \mathscr{D}_{\frac{1}{2}}^+ \oplus 2\mathscr{D}_1^+ \oplus 2\mathscr{D}_{\frac{3}{2}}^+ \oplus \cdots. \tag{7}$$

Assuming that mode a has more photons than mode b, an arbitrary two-mode state $|n_a, n_b\rangle$, is equivalent to SU(1,1) weight states $|k,\mu\rangle$ with

$$k = \frac{n_a - n_b + 1}{2}, \qquad \mu = \frac{n_a + n_b + 1}{2}. \tag{8}$$

As the irrep label k is proportional to the photon number difference, and irrep spaces are invariant under SU(1,1), the photon number difference is conserved for two-mode squeezers. In other words, the photon number difference in the two modes entering these optical elements equals the photon number difference leaving them.

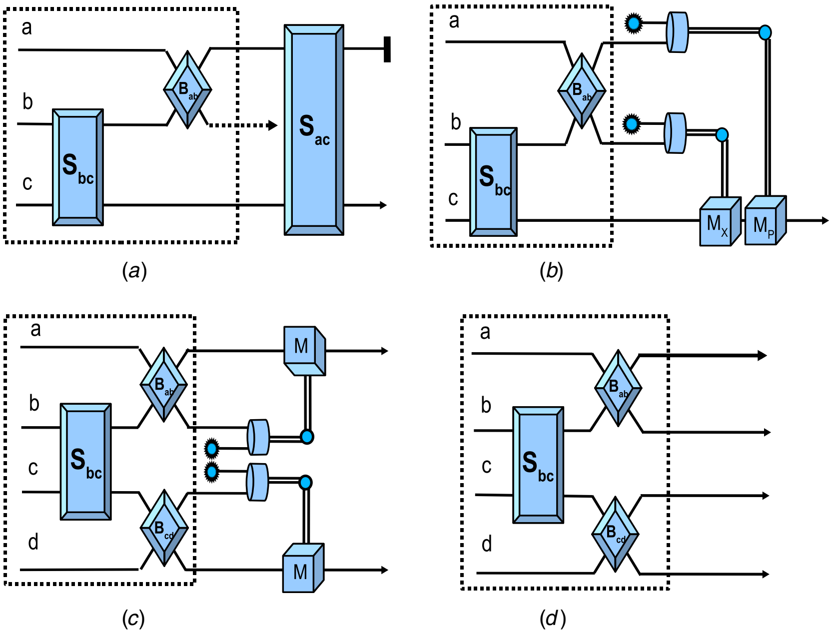

Figure 1. The dashed rectangles show that how three-mode or four-mode squeezed states, which are a key source of entanglement, can be generated by distributing a two-mode squeezed state over other modes. Such states are a required ingredient for various quantum information tasks, such as (*a*) quantum state sharing [22], (*b*) quantum teleportation [11], (*c*) entanglement swapping [5] and (*d*) tests of local realism and the Einstein–Podolsky–Rosen paradox [6].

Discovering an appropriate realization of a symmetry group enables a clear understanding of a system via the mathematical properties that are already known for these groups. We can exploit representation-theoretic machinery, such as selection rules and branching rules to facilitate calculations [17, 24, 25]. Output states are then characterized by generalized coherent states that are known for groups such as SU(1,1) [26].

Often, understanding the symmetry of a system also brings with it much needed physical insight as systems become more complicated. However, we will limit the scope of this paper to identifying that symmetry and constructing representations for a class of linear optical networks plus squeezers that occur in several important optical quantum information protocols as described in the following section.

3. Motivation: three-mode squeezing and the SU(1,1) Lie group

As mentioned in the introduction, the motivation for this research stems from recent optical quantum information experiments. In figure 1 we have included simple schematics of these experiments. As is indicated in the figure, all these systems incorporate a two-mode squeezer with one (or both) of the squeezer's output modes mixed with a third (and fourth) mode on a 50:50 beam splitter, thereby distributing entanglement across three (or four) modes in the network; see figure 2.

Two-mode squeezing action is given in equation (1); the beamsplitter action on modes a and b is given by

$$\hat{B}_{ab}(\theta, \phi) = \exp[\theta(\hat{a}^\dagger \hat{b}\, \mathrm{e}^{\mathrm{i}\phi} - \hat{a}\hat{b}^\dagger\, \mathrm{e}^{-\mathrm{i}\phi})/2]. \tag{9}$$

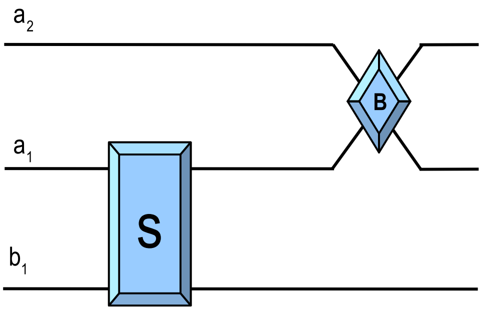

Figure 2. The 'primitive' three-mode optical network in which tripartite entangled states are produced. Such states are the first step towards multipartite entangled states and are applied in various optical quantum schemes some of which are shown in figure 1.

For the given values of the arguments, the transformation performed by the three-mode component of figure 2 is [9]

$$\hat{B}^{+}_{a_2a_1}\hat{S}_{a_1b_1}(2\mathrm{i}\eta) = \hat{S}_{a_2a_1b_1}(\sqrt{2}\mathrm{i}\eta)\hat{B}^{+}_{a_2a_1} \tag{10}$$

for

$$\hat{B}^{\pm}_{ab} \equiv \hat{B}_{ab}\left(\frac{\pi}{2}, \pm\pi\right) \tag{11}$$

and

$$\hat{S}_{a_2a_1b_1}(\sqrt{2}\mathrm{i}\eta) = \exp\left[\frac{-\mathrm{i}\eta}{\sqrt{2}}(\hat{a}_1\hat{b}_1 - \hat{a}_2\hat{b}_1) + \frac{\mathrm{i}\eta^*}{\sqrt{2}}\left(\hat{a}_1^{\dagger}\hat{b}_1^{\dagger} - \hat{a}_2^{\dagger}\hat{b}_1^{\dagger}\right)\right]. \tag{12}$$

It is not difficult to check that the generators of this transformation satisfy a $\mathfrak{su}(1,1)$ algebra with ladder operator

$$\hat{K}_{+} = \tfrac{1}{\sqrt{2}}\left(\hat{a}_1^{\dagger}\hat{b}_1^{\dagger} - \hat{a}_2^{\dagger}\hat{b}_1^{\dagger}\right). \tag{13}$$

The specifics of this realization have been discussed in our preliminary investigation [27]. In the following section, we generalize this result to arbitrarily many modes.

4. Results: multimode squeezing and the SU(1,1) Lie group

In this section, we show that the three-mode entanglement distributing component given in figure 2 can be extended to arbitrarily many modes while still being generated by a $\mathfrak{su}(1,1)$ algebra. We then analyze the representations of this algebra on the multimode Fock space, giving the new basis of SU(1,1) weight states.

We are considering an optical network (see figure 3) that comprises one two-mode squeezer in which one output state is mixed via beam splitters between r modes, created by $\left\{\hat{a}_l^{\dagger}\right\}_{l=1}^{r}$, and the other mixed with s modes, created by $\left\{\hat{b}_l^{\dagger}\right\}_{l=1}^{s}$. For simplicity, we consider only 50:50 beam splitters with beam splitters for the upper r modes having a phase $\phi = \pi$ and those for the lower s modes having $\phi = -\pi$. Based on the Baker–Campbell–Hausdorff formula[1], the resulting transformation performed by the multimode network is

$$\hat{B}^{+}_{a_ra_{r-1}}\cdots\hat{B}^{+}_{a_2a_1}\hat{B}^{-}_{b_sb_{s-1}}\cdots\hat{B}^{-}_{b_2b_1}\hat{S}_{a_1b_1}(\eta) = \hat{S}_{A_rB_s}(\eta)\hat{B}^{+}_{a_ra_{r-1}}\cdots\hat{B}^{+}_{a_2a_1}\hat{B}^{-}_{b_sb_{s-1}}\cdots\hat{B}^{-}_{b_2b_1}, \tag{14}$$

[1] The Baker–Campbell–Hausdorff formula is $\mathrm{e}^{\mathrm{i}\lambda\hat{A}}\hat{B}\,\mathrm{e}^{-\mathrm{i}\lambda\hat{A}} = \hat{B} + (\mathrm{i}\lambda)[\hat{A},\hat{B}] + \frac{(\mathrm{i}\lambda)^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{(\mathrm{i}\lambda)^3}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \cdots$.

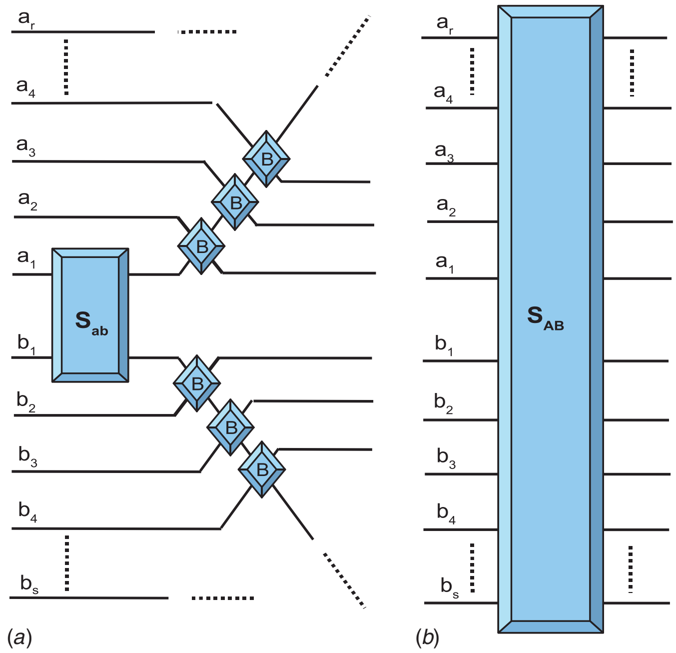

Figure 3. Equivalence between a typical multimode quantum optical network and a pseudo-two-mode squeezer. (*a*) A multimode optical network comprising a two-mode squeezer (S_{ab}) and several 50:50 beam splitters. (*b*) A pseudo-two-mode squeezer (S_{AB}).

with $\hat{S}_{A_rB_s}$ being a multimode squeezing operator (see figure 3). This operator can be viewed as a Bogoliubov-transformed two-mode squeezer and is given by

$$\hat{S}_{A_rB_s} = \exp\left[-\mathrm{i}\left(\eta\hat{A}_r\hat{B}_s + \eta^*\hat{A}_r^\dagger\hat{B}_s^\dagger\right)/2\right], \tag{15}$$

where $\hat{A}_r$ and $\hat{B}_s$ are the generalized boson (*pseudo-boson*) operators

$$\hat{A}_r = \sum_{l=1}^{r-1}\frac{(-1)^{l-1}\hat{a}_l}{\sqrt{2^l}} + \frac{(-1)^{r-1}\hat{a}_r}{\sqrt{2^{r-1}}}, \qquad \hat{B}_s = \sum_{l=1}^{s-1}\frac{(-1)^{l-1}\hat{b}_l}{\sqrt{2^l}} + \frac{(-1)^{s-1}\hat{b}_s}{\sqrt{2^{s-1}}} \tag{16}$$

given in terms of the original optical modes a_l and b_l for $r, s \geqslant 2$. Of course, these pseudo operators satisfy the canonical bosonic commutation relations

$$[\hat{A}_r, \hat{A}_r^\dagger] = \hat{\mathbb{1}}, \qquad [\hat{B}_s, \hat{B}_s^\dagger] = \hat{\mathbb{1}}. \tag{17}$$

The Hamiltonian generating $\hat{S}_{A_rB_s}$ is a linear combination of the two operators $\hat{A}_r\hat{B}_s$ and $\hat{A}_r^\dagger\hat{B}_s^\dagger$. These together with their commutators are closed under the commutation relations of a $\mathfrak{su}(1,1)$ algebra and thus provide a pseudo-two-boson realization of $\mathfrak{su}(1,1)$

$$\hat{K}_- = \hat{A}_r\hat{B}_s, \qquad \hat{K}_+ = \hat{A}_r^\dagger\hat{B}_s^\dagger, \qquad \hat{K}_0 = \tfrac{1}{2}\left(\hat{A}_r^\dagger\hat{A}_r + \hat{B}_s^\dagger\hat{B}_s + \hat{\mathbb{1}}\right). \tag{18}$$

The Casimir operator for this realization is

$$\hat{K}^2 = \tfrac{1}{4}\left[\left(\hat{A}_r^\dagger\hat{A}_r - \hat{B}_s^\dagger\hat{B}_s\right)^2 - \hat{\mathbb{1}}\right] \tag{19}$$

with $\hat{A}_r^\dagger\hat{A}_r$ and $\hat{B}_s^\dagger\hat{B}_s$ the pseudo-number operators, $\hat{N}_A$ and $\hat{N}_B$, respectively. If we define pseudo-number states such that

$$\hat{N}_A|n_A\} = n_A|n_A\}, \qquad \hat{N}_B|n_B\} = n_B|n_B\}, \tag{20}$$

then the entire situation is formally equivalent to the two-mode realization given in section 2. From equation (8), we can read off the labels for a basis of SU(1,1) weight states $|k, \mu\rangle$ in terms of the pseudo-number states $|n_A n_B\}$:

$$\begin{array}{ll} k = \frac{1}{2}(|n_A - n_B| + 1), & \mu = k + m = \frac{1}{2}(n_A + n_B + 1), \\ k = \frac{1}{2}, 1, \frac{3}{2}, \ldots, & m = 0, 1, 2, \ldots, \\ n_A = n_{a_1} + n_{a_2} + \cdots + n_{a_r}, & n_B = n_{b_1} + n_{b_2} + \cdots + n_{b_s}. \end{array} \tag{21}$$

Therefore, the difference between the total photon number for modes a_1 to a_r and the total photon number for modes b_1 to b_s is conserved.

All that remains in order to specify the new basis is to give the pseudo-number states in terms of the original modes. For $|n_A\}$, and analogously for $|n_B\}$, we obtain

$$|n_A\} = \frac{(A_r^\dagger)^{n_A}}{\sqrt{n_A!}}|0\} = \sum_{\mathbf{n}} C_{\mathbf{n}}^{n_A} |\mathbf{n}\rangle, \tag{22}$$

where $\mathbf{n} = (n_{a_1}, n_{a_2}, \ldots, n_{a_r})$ is a partition of n_A:

$$\sum_{l=1}^{r} n_{a_l} = n_A, \qquad 0 \leqslant n_{a_l} \in \mathbb{Z}, \tag{23}$$

and $|\mathbf{n}\rangle = |n_{a_1}, n_{a_2}, \ldots, n_{a_r}\rangle$ is a multimode number state. From equations (16) and (22), we have

$$C_{\mathbf{n}}^{n_A} = \left(\frac{n_A!}{n_{a_1}! n_{a_2}! \cdots n_{a_r}!}\right)^{1/2} \left[\prod_{l=1}^{r-1} \left(\frac{(-1)^{l-1}}{\sqrt{2^l}}\right)^{n_{a_l}}\right] \left(\frac{(-1)^{r-1}}{\sqrt{2^{r-1}}}\right)^{n_{a_r}}. \tag{24}$$

It is not difficult to verify that these states are orthonormal, $\{n_A|m_A\} = \delta_{n_A m_A}$.

Take for example the experimentally significant three-mode case depicted in figure 2. In this case, the pseudo-number states, in terms of the original modes a_1 and a_2, are given by

$$|n_A\} = \frac{1}{\sqrt{2^{n_A}}} \sum_{n_{a_1}=0}^{n_A} (-1)^{N_A - n_{a_1}} \binom{n_A}{n_{a_1}}^{1/2} |n_{a_1} n_{a_2}\rangle, \qquad n_A = n_{a_1} + n_{a_2}, \tag{25}$$

which is normalized because

$$\frac{1}{2^{n_A}} \sum_{n_1=0}^{n_A} \binom{n_A}{n_{a_1}} = 1. \tag{26}$$

Assuming that the number of photons in modes a_1 and a_2 exceeds the photon number in mode b_1, the μth three-mode SU(1,1) weight state in irrep k, in terms of these three photon number states, is

$$|k, \mu\rangle = \frac{1}{\sqrt{2^{k+\mu-1}}} \sum_{n_{a_1}=0}^{k+\mu-1} (-1)^{n_{a_1}} \binom{k+\mu-1}{n_{a_1}}^{1/2} |k + \mu - n_{a_1} - 1, n_{a_1}, \mu - k\rangle. \tag{27}$$

However, by this assumption we can only get half of the irreps. To get the other half, one needs to exchange $k + \mu - 1$ and $\mu - k$ in the above expression.

5. Discussion

This SU(1,1) basis enables us to use a variety of mathematical properties that have already been established for this group, thereby significantly facilitating calculations. For instance, the SU(1,1) Clebsch–Gordan coefficients [24, 28, 29] are known for coupling different

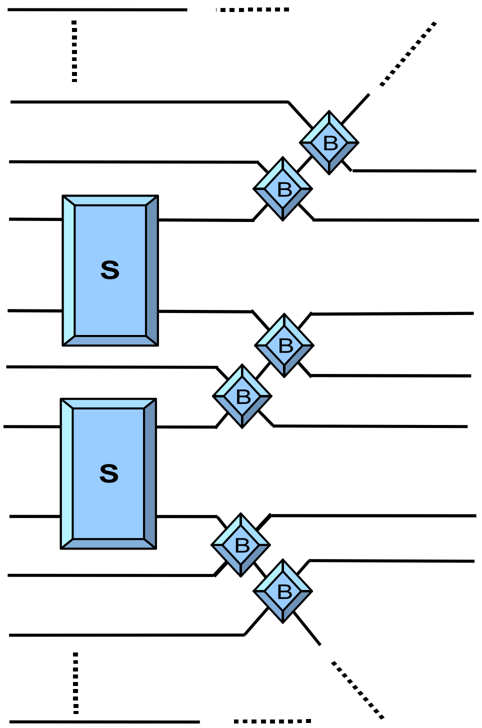

Figure 4. A multimode network with more than one two-mode squeezer. Such a network can be built by concatenating different parallel sections each being of the type of multimode network considered earlier. The known SU(1,1) Clebsch–Gordan and Racah coefficients can be used to couple the corresponding representations and to obtain the resultant states.

representations and obtaining the resultant states. These coefficients are very handy if one wants to concatenate some of these typical multimode networks 'in parallel' (see figure 4). Furthermore, if the concatenation takes place between more than two such optical networks, then the Racah coefficients [30] give the transformations between different orderings of the couplings. The Wigner–Eckart theorem [18] makes the calculation of matrix elements of a tensor operator, or any operator generated from the elements of the algebra (a linear combination thereof, such as the squeezing operator) in this SU(1,1) basis much simpler than a direct approach. Generalized coherent states [26], orthogonality and asymptotic behavior of the matrix elements [17] are some other examples of the well-known properties for this group.

The significance of our result is that it allows for arbitrary input states, in contrast to existing methods which usually rely upon Gaussian inputs. This technique can therefore be used to characterize a larger class of output states. Moreover, because the algebraic structure is independent of the number of modes, complicated multimode linear quantum optical networks with two-mode squeezing, that are of this form, can be dealt with much more ease than with, for example, Wigner functions, in which one needs to calculate $2n$ integrals, or covariance matrices for which $4n^2$ matrix elements must be calculated, where n is the number of modes.

These advantages come with a cost, namely a multiplicity of SU(1,1) irreps that grows as the number of modes. For $r + s$ modes there are always only two pseudo-boson operators, $\hat{A}_r$ and $\hat{B}_s$, leaving $r+s-2$ 'directions' in Fock space. Finding $r+s-2$ orthogonal pseudo-boson operators and using them to label multiplicities would be tedious but not impossible for large r or s: this is an open question for further research. Also, we have limited our attention to linear quantum optical networks with two-mode squeezing with the specific sequence of beamsplitters given by equation (14)—we considered this form since it is a direct extension of the networks in existing experiments. This SU(1,1) realization will occur for any Bogoliubov transformation that mixes each output port of the two-mode squeezer separately; that is, any Bogoliubov transformation that does not mix a modes and b modes (see figure 3). When these modes are mixed 'in series' by some multiport optical element a far more complicated situation arises, as exemplified in the following section.

6. More complicated scenarios

Consider the network arising from the reconstruction protocol of the quantum secret sharing experiment in figure 1. Here, there is an extra two-mode squeezer operating on one mode from the upper set and one mode from the lower set of the three-mode squeezing scheme given in figure 2. Consequently, we obtain the identity

$$\hat{S}_{a_2b_1}(\eta')\hat{S}_{a_2a_1b_1}(\sqrt{2}\mathrm{i}\eta) = \hat{\mathcal{S}}_{a_2a_1b_1}(\eta,\eta')\hat{S}_{a_2b_1}(\eta'), \tag{28}$$

where $\hat{\mathcal{S}}_{a_2a_1b_1}(\eta,\eta')$ is the exponential of a linear combination of 12 terms. However, by fixing two of the parameters,

$$\phi' = \frac{\pi}{2} \Rightarrow \eta' = s'\exp(\mathrm{i}\phi') = \mathrm{i}s', \qquad \eta = \eta* \Rightarrow \phi = 2m\pi \tag{29}$$

for $m = 0, 1, 2, \ldots,$ we obtain

$$\hat{\mathcal{S}}_{a_2a_1b_1}(\eta,\eta') = \exp\left[\frac{\eta}{\sqrt{2}}\cosh\left(\frac{s'}{2}\right)\left(\hat{a}_1\hat{b}_1 - \hat{a}_1^\dagger\hat{b}_1^\dagger\right) + \frac{\eta}{\sqrt{2}}\sinh\left(\frac{s'}{2}\right)\left(\hat{a}_2^\dagger\hat{a}_1 - \hat{a}_2\hat{a}_1^\dagger\right) - \frac{\eta}{\sqrt{2}}\left(\hat{a}_2\hat{b}_1 - \hat{a}_2^\dagger\hat{b}_1^\dagger\right)\right]. \tag{30}$$

This operator is generated by

$$\hat{K}_x = -\mathrm{i}\left(\hat{a}_1\hat{b}_1 - \hat{a}_1^\dagger\hat{b}_1^\dagger\right), \qquad \hat{K}_y = -\mathrm{i}\left(\hat{a}_2\hat{b}_1 - \hat{a}_2^\dagger\hat{b}_1^\dagger\right), \qquad \hat{K}_z = \mathrm{i}\left(\hat{a}_2^\dagger\hat{a}_1 - \hat{a}_2\hat{a}_1^\dagger\right), \tag{31}$$

which satisfy the commutation relations of $\mathfrak{su}(1,1)$. A unitarily equivalent form of these operators has been found previously by Abdalla *et al* [32]. One can unitarily transform these operators using

$$\hat{U} = \exp\left[\mathrm{i}\frac{\pi}{4}\left(\hat{a}_2^\dagger\hat{a}_1 + \hat{a}_2\hat{a}_1^\dagger\right)\right], \tag{32}$$

in order to diagonalize $\hat{K}_z$ in the Fock basis:

$$\hat{\hat{K}}_z = \hat{U}\hat{K}_z\hat{U}^\dagger = \hat{a}_2^\dagger\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_1. \tag{33}$$

However, this transformation does not simplify the form of the other operators; in particular, the Casimir operator

$$\hat{\hat{K}}^2 = \hat{U}\hat{K}^2\hat{U}^\dagger = \left(\hat{a}_2^\dagger\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_1\right)^2 - 2\left(\hat{a}_2^\dagger\hat{a}_2 + \hat{a}_1^\dagger\hat{a}_1 + \hat{\mathbb{1}}\right)\hat{b}_1^\dagger\hat{b}_1 - \left(\hat{a}_2^\dagger\hat{a}_2 + \hat{a}_1^\dagger\hat{a}_1 + 2\hat{\mathbb{1}}\right) + 2\mathrm{i}\left(\hat{a}_2^\dagger\hat{a}_1^\dagger\hat{b}_1^{\dagger 2} - \hat{a}_2\hat{a}_1\hat{b}_1^2\right). \tag{34}$$

Identifying the appropriate representations of this realization of $\mathfrak{su}(1,1)$ carried by the Fock space of three modes is quite challenging. For one, the lowering operator $\hat{K}_-$ is a linear combination of both lowering *and* raising operators. This means that the lowest weight $\mu = k$ states are not Fock states but superpositions of Fock states, a situation that does not occur in the standard one- and two-mode realizations. More importantly, $\hat{K}_z$ can have both positive and negative eigenvalues in the Fock basis, which, along with the complicated structure of $\hat{K}^2$, suggests that we have left the realm of discrete SU(1,1) irreps and must consider the less intuitive continuous irreps that do not have extremal weights. The problem of which classes of SU(1,1) irreps are supported on multimode Fock spaces is an interesting one deserving further study. Understanding these representations would give us a powerful analytical tool since it would give a complete symmetry adapted basis for these and ultimately more complicated optical networks, which would aid in the investigation of squeezing and entanglement therein.

7. Conclusions

We have established a novel basis lying in the discrete representation of SU(1,1). This basis is adapted to physical problems of multimode squeezing. Such problems occur in various experimental setups for optical quantum information schemes involving a two-mode squeezer and several passive optical elements. This technique facilitates the calculation of output states without restricting to 'standard' input states like Gaussians.

We also showed that some interesting mathematical problems arise in more complicated multimode schemes. These problems seem inherent to linear quantum optical networks with squeezers and with more than two mixed modes; e.g., it also arises in a four-mode realization of squeezing studied by Bartlett *et al* [32]. This interesting complication opens questions about whether, by concatenating optical networks, each with just one two-mode squeezer, the description remains within a discrete series representation of SU(1,1). Based on the evidence discussed in the previous section, it seems unlikely. Such problems are important in considering practical quantum information tasks, in studying squeezing as a resource and in similar problems; our work reported here is an important step in this direction.

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